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Relevant Definitions

Definition 4.2.4 $\mathcal{S}(\mathbb{R}^n)$ is the linear space of $\phi \in C^\infty(\mathbb{R}^n)$ satisfying

$$\lim_{|x|\to\infty} |x|^k |D^\alpha \varphi(x)| = 0$$

for all $k = 0, 1, 2, \ldots$ and all α. $\S(\mathbb{R}^n)$ is called the Schwartz space.

Definition 4.2.38 Let $\varphi \in C^\infty(\mathbb{R})$. For each pair of integers $k \geq 0$, $m > 0$, set

$$p_{m,k}(\varphi) \equiv \max_{0\leq j\leq k} \sup_{-m\leq x\leq m} |D^j \varphi(x)|.$$

Corollary 4.2.47 A sequence φ_j of function in $C^\infty(\mathbb{R}^n)$ converges to φ if, for every m, k

$$\lim_{j\to\infty} p_{m,k}(\varphi_j - \varphi) = 0$$

Definition 4.2.60 For $\varphi \in \mathcal{S}(\mathbb{R}^n)$ set

$$q_{m,k}(\varphi) \equiv \max_{|\alpha|\leq k} \sup_{x\in\mathbb{R}^n} |x|^m |D^\alpha \varphi(x)|.$$

Definition 4.2.61 A sequence φ_j is said to converge to φ in $\mathcal{S}(\mathbb{R}^n)$ if

$$\lim_{j\to\infty} q_{m,k}(\varphi_j - \varphi) = 0$$

for all $m, k \in \mathbb{N} \cup \{0\}$.

Definition 4.2.51 A sequence φ_j is said to converge to 0 in $C_0^\infty(\mathbb{R}^n)$ if

1. there is an R such that $\varphi_j(x) = 0$ when $|x| > R$, for all j,and
2. for each k, the seminorms $p_{R,k}(\varphi_j)$ approach 0 as $j \to \infty$.

1. Show that the family of semi-norms

$$\zeta_{m,k}(\psi) = \sup_{x\in\mathbb{R}^n} \max_{0\leq|\alpha|\leq k} (1 + ||x||^m) |D^\alpha(\psi(x))|$$

generates the usual topology on the Schwartz space $\mathcal{S}(\mathbb{R}^n)$.

Solution. To show that the two topologies are equivalent requires convergence in one iff convergence in the second. Consider the "standard" family of semi-norms on $\mathcal{S}(\mathbb{R}^n)$,

$$q_{m,k}(\psi) = \sup_{x\in\mathbb{R}^n} \max_{0\le|\alpha|\le k} ||x||^m |D^\alpha(\psi(x))|.$$

Showing convergence in one implies the second, and vice-versa, requires placing a bound on one family of semi-norms in terms of a linear combination of the second family of semi-norms.

I will first show that I can bound $q_{m,k}$ in terms of $\zeta_{m,k}$. Fix $m, k \in \mathbb{N}$. This direction is easy, as by definition $q_{m,k}(\phi) \le \zeta_{m,k}(\phi)$. The second direction follows immediately, since $\zeta_{m,k}(\phi) \le q_{m,k}(\phi) + 2q_{0,k}(\phi)$.

$\therefore$ $\zeta_{m,k}$ generates the same topology as $q_{m,k}$.

2. Problems 4.2.62 and 4.2.63, Pg. I-216 in the book.

   (4.2.62) Let $\varphi \in \mathcal{S}(\mathbb{R})$ and define

   $$\varphi_j(x) = \frac{1}{j}\varphi\big(\frac{x}{j}\big).$$

   Does $\varphi \to 0$ according to Definition 4.2.60?

   Solution.

   Definition. (4.2.60) For $\phi \in \mathcal{S}(\mathbb{R}^n)$, set

   $$q_{m,k}(\phi) \stackrel{\text{def}}{=} \max_{|\alpha|\le k} \sup_{x\in\mathbb{R}^n} |x|^m |D^\alpha \phi(x)|.$$

   A sequence is said to converge to ϕ in $\mathcal{S}(\mathbb{R}^n)$, if

   $$\lim_{j\to\infty} q_{m,k}(\phi_j - \phi) = 0, \quad \forall m, k = 0, 1, \dots$$

   To show that ϕ_j converges to 0, according to Def. (4.2.60), I must show that $q_{m,k}(\phi_j) \to 0$, for all m, k. So then

   $$\begin{aligned}
   \lim_{j\to\infty} q_{m,k}(\phi_j) &= \lim_{j\to\infty} \max_{|\alpha|\le k} \sup_{x\in\mathbb{R}} |x|^m |D^\alpha \phi_j(x)| \\
   &= \lim_{j\to\infty} \max_{|\alpha|\le k} \sup_{x\in\mathbb{R}} |x|^m |D^\alpha \frac{1}{j}\phi(x/j)| \\
   &= \lim_{j\to\infty} \max_{|\alpha|\le k} \sup_{x\in\mathbb{R}} |x|^m \frac{1}{j^{\alpha+1}} |\phi^{(\alpha)}(x/j)| \\
   &= \lim_{j\to\infty} \max_{|\alpha|\le k} \sup_{x\in\mathbb{R}} \frac{1}{j^{\alpha+m+1}} |jx|^m |\phi^{(\alpha)}(x/j)|
   \end{aligned}$$

But, since $\max_{|\alpha|\leq k} \sup_{x\in\mathbb{R}} |jx|^m |\phi^{(\alpha)}(xj)|$ is bounded, since $\phi \in \mathcal{S}(\mathbb{R})$, then we have that

$$\lim_{j\to\infty} q_{m,k}(\phi_j) = 0 \tag{1}$$

Therefore we see in Eqn. 1 that ϕ_j indeed will converge to 0, according to the above definition.

(4.2.63) Let $\varphi \in \mathcal{S}(\mathbb{R})$. Set $\varphi_j = \varphi/j$. Does this sequence converge in $\mathcal{S}(\mathbb{R})$? In $C_0^\infty(\mathbb{R})$? Set $\varphi_j = \varphi(jx)/j$. Does this sequence converge in $\mathcal{S}(\mathbb{R})$?

Solution. First we show that ϕ_j will converge in $\mathcal{S}(\mathbb{R})$.

$$\begin{aligned}
\lim_{j\to\infty} q_{m,k}(\phi_j) &= \lim_{j\to\infty} \max_{|\alpha|\leq k} \sup_{x\in\mathbb{R}} |x|^m |D^\alpha \phi_j(x)| \\
&= \lim_{j\to\infty} \max_{|\alpha|\leq k} \sup_{x\in\mathbb{R}} |x|^m |D^\alpha \frac{1}{j} \phi(x)| \\
&= \lim_{j\to\infty} \max_{|\alpha|\leq k} \sup_{x\in\mathbb{R}} \frac{1}{j} |x|^m |\phi^{(\alpha)}(x)|
\end{aligned}$$

But since $\phi \in \mathcal{S}(\mathbb{R})$, we know that

$$\max_{|\alpha|\leq k} \sup_{x\in\mathbb{R}} |x|^m |\phi^{(\alpha)}(x)| < \infty$$

Hence as $j \to \infty$, we have that $q_{m,k} \to 0$, so we have that ϕ_j will converge to 0 in $\mathcal{S}(\mathbb{R})$, according to Def. (4.2.60).

To show that ϕ_j does not converge in $C_0^\infty(\mathbb{R})$, we simply look at some $a \in \mathbb{R}$ such that $\phi(a) \neq 0$. Then $\phi_j \neq 0$, which contradicts a necessary condition for a sequence to converge in C_0^∞ which states that ϕ_j must vanish for all j, when x is sufficiently large(*independent of* j).

Finally, we wish to consider the function $\varphi_j(x) = \phi(jx)/j$. I note here that

$$|D^\alpha \varphi_j = j^{\alpha-1} |\phi^{(\alpha)}(jx)|$$

then we have

$$\begin{aligned}
&\lim_{j\to\infty} \max_{\alpha\leq k} \sup_{x\in\mathbb{R}} |x|^m j^{\alpha-1} |\phi^{(\alpha)}(jx)| \\
&\qquad\qquad = \lim_{j\to\infty} \max_{x\in\mathbb{R}} j^{k-m-1} |jx|^m |\phi^{(k)}(jk)|
\end{aligned}$$

and since $\phi \in \mathcal{S}(\mathbb{R})$, we know that $|jx|^m |\phi^{(k)}(jk)|$ is bounded, hence φ will not converge to 0 if $k > m + 1$.

3. Problem 4.2.64, Pg. I-216 in the book.

   (4.2.64) Let p be a polynomial of one variable, and let $\varphi \in \mathcal{S}(\mathbb{R})$. Show that $p\varphi \in \mathcal{S}(\mathbb{R})$. If $\varphi_j \to \varphi$ in the sense of Definition 4.2.61, what can one say about the convergence of the sequence $p\varphi_j$?

Solution. We will show that $p\phi_j$ will converge, using induction. First we consider a first degree polynomial, $p(x) = ax$, where $a < \infty$. To show convergence, we must show that $p\phi$ converges according to Def. (4.2.60). So

$$\begin{aligned}
\lim_{j\to\infty} q_{m,k}(p\phi_j) &= \lim_{j\to\infty} \max_{|\alpha|\le k} \sup_{x\in\mathbb{R}} |x|^m |D^\alpha p\phi_j(x)| \\
&= \lim_{j\to\infty} \max_{|\alpha|\le k} \sup_{x\in\mathbb{R}} |x|^m |D^\alpha ax\phi_j(x)| \\
&= \lim_{j\to\infty} \max_{|\alpha|\le k} \sup_{x\in\mathbb{R}} |x|^m |ax\phi^{(\alpha)}(x) + \alpha a\phi^{(\alpha-1)}(x)| \\
&= aq_{m+1,\alpha} + a\alpha q_{m,\alpha-1}
\end{aligned}$$

But since $\phi \in \mathcal{S}(\mathbb{R})$, we have that that $q_{m,k} < \infty$ for all $m, k = 0, 1, \ldots$. Therefore the sum above is finite, given that a, α themselves are finite, and we have that for a first degree polynomial, $p\phi$ will converge.

Next, we will assume that $p\phi$ converges for an n^{th} degree polynomial p, and show that it will converge for an $(n+1)^{th}$ degree polynomial. In the analysis that follows, I ignore all lower degree polynomials, since they are assumed to converge. Again, let $a \geq 0$

$$\begin{aligned}
\lim_{j\to\infty} q_{m,k}(p\phi_j) &= \lim_{j\to\infty} \max_{|\alpha|\le k} \sup_{x\in\mathbb{R}} |x|^m |D^\alpha p\phi_j(x)| \\
&= \lim_{j\to\infty} \max_{|\alpha|\le k} \sup_{x\in\mathbb{R}} |x|^m |D^\alpha ax^{n+1}\phi_j(x)| \\
&\le \lim_{j\to\infty} \max_{|\alpha|\le k} \sup_{x\in\mathbb{R}} |x|^m \sum_{i=0}^{\alpha} a_i |x^{n+1-i}\phi_j^{(\alpha-i)}|
\end{aligned}$$

where if $n + 1 < i$, then let $x^{n+1-i} \stackrel{def}{=} 1$, and all constants are absorbed into the a_i's. Then

$$\begin{aligned}
\lim_{j\to\infty} \max_{|\alpha|\le k} \sup_{x\in\mathbb{R}} |x|^m \sum_{i=0}^{\alpha} a_i |x^{n+1-i}\phi_j^{(\alpha-i)}| &= \lim_{j\to\infty} \max_{|\alpha|\le k} \sup_{x\in\mathbb{R}} \sum_{i=0}^{\alpha} a_i |x|^{m+n+1-i} |\phi_j^{(\alpha-i)} \\
&= \sum_{i=0}^{\alpha} q_{m+1-i,\alpha-i}
\end{aligned}$$

But since each of the $q_{m,k} < \infty$, we have that the sum is finite also. Thus, we have that $p\phi_j$ converges, when p is an $(n+1)^{th}$ degree ploynomial. Therefore we have that, given that p is a polynomial, and that $\phi_j \in \mathcal{S}(\mathbb{R})$, then $p\phi_j$ converges according to Def. (4.2.60).

4. Problems 4.2.67 and 4.2.69, Pg. I-217 in the book.

   (4.2.67) Prove that PV $\frac{1}{x}$ is a continuous linear function on $\mathcal{S}(\mathbb{R})$.

Solution. To show that $\mathrm{PV}\frac{1}{x}$ is a continuous linear function on $\mathcal{S}(\mathbb{R})$, we essentially only need to show that $\mathrm{PV}\frac{1}{x}$ is sequentially continuous, since showing linearity is trivial. Here we will make use of $q_{m,k}$ as defined in Def (4.2.60).

Suppose that there exists a sequence $\phi_n \in \mathcal{S}(\mathbb{R})$ such that $\phi_n \to \phi$, $\phi \in \mathcal{S}(\mathbb{R})$. Without loss of generality, let $\phi_n \to 0$ for the purposes of this computation. Then we have that for $\mathrm{PV}\frac{1}{x}(\phi_n)$, for $n \in \mathbb{N}$ there exists some $\epsilon > 0$ such that

$$\begin{aligned}
\mathrm{PV}\tfrac{1}{x}(\phi_n) &= \lim_{\epsilon\to 0}\int_{|x|>\epsilon}\frac{\phi_n(x)}{x}dx \\
&= \int_{-1}^{1}\psi_n(x)dx + \int_{|x|>1}\frac{\phi_n(x)}{x}dx \text{ where } \psi \text{ is defined to be} \\
\psi_n(x) &= \begin{cases} \frac{\phi_n(x)-\phi_n(0)}{x}, & x \neq 0 \\ \phi_n'(x), & x = 0 \end{cases}
\end{aligned}$$

Then

$$\left|\mathrm{PV}\tfrac{1}{x}(\phi_n)\right| \leq \int_{-1}^{1}|\psi_n(x)|dx + \int_{|x|>1}\left|\frac{\phi_n(x)}{x}\right|dx \tag{2}$$

Now we also know from the definition of $q_{m,k}$ above, that

$$q_{0,1}(\phi) = \max_{|\alpha|\leq 1}\sup_{x\in\mathbb{R}}|D^\alpha\phi(x)|, \qquad q_{1,0}(\phi) = \sup_{x\in\mathbb{R}}|x|\phi(x)$$

Hence

$$\begin{aligned}
|\phi_n'(x)| &\leq q_{0,1}(\phi_n), \text{ and that} \\
|x\phi_n(x)| &\leq q_{1,0}(\phi_n)
\end{aligned}$$

Inserting the inequalities into Eqn 2, we have that

$$\begin{aligned}
\left|\mathrm{PV}\tfrac{1}{x}(\phi_n)\right| &\leq \int_{-1}^{1}q_{0,1}(\phi_n)dx + \int_{|x|>1}\frac{q_{1,0}(\phi_n)}{x^2}dx \\
&= 2q_{0,1}(\phi_n) + 2q_{1,0}(\phi_n)
\end{aligned}$$

But, since $\phi_n \to 0$, we have that $q_{m,k}(\phi_n) \to 0$, forall m, k. Hence we have that

$$\mathrm{PV}\tfrac{1}{x}(\phi_n) \to 0, \text{ given that } \phi_n \to 0$$

Which, of course, implies that $\mathrm{PV}\frac{1}{x}(\phi_n)$ is continuous on $\mathcal{S}(\mathbb{R})$. Therefore I have shown that $\mathrm{PV}\frac{1}{x}(\phi_n)$ is a continuous, linear function on $\mathcal{S}(\mathbb{R})$.

(4.2.69) For $\varphi \in \mathcal{S}(\mathbb{R})$, define

$$T(\varphi) = \sum_{n=-\infty}^{\infty}\varphi(n).$$

Is this a distribution (i.e., a continuous linear function)? (If it is a distribution, it would be written

$$\sum_{n=-\infty}^{\infty} \delta(x-n).)$$

Solution. Consider the following, which uses the fact that $\phi \in \mathcal{S}(\mathbb{R})$

$$\begin{aligned} q_{2,0}(\phi_j) &= \sup_{x\in\mathbb{R}} |x|^2|\phi_j| \\ &< \infty \\ &\Downarrow \\ n^2|\phi(n)| \leq q_{2,0}(\phi(n)) & \\ |\phi(n)| \leq \frac{q_{2,0}(\phi(n))}{n^2} & \end{aligned}$$

Which is clearly bounded.

Linearity is obvious here, and we can use the bound above to establish continuity.

$$\begin{aligned} T(\phi_j) - T(\phi) &= T(\phi_j - \phi), \text{ exploiting the linearity in } T \\ &= \sum_n (\phi_j(n) - \phi(n)) \\ &\leq \sum_n \frac{1}{n^2}(q_{2,0}(\phi) - q_{2,0}(\phi)) \\ &= 0 \end{aligned}$$

Hence we have that $\phi_j \to \phi$ implies that $T(\phi_j) \to T(\phi)$. Therefore we have that

$$T(\phi) = \sum_{n=-\infty}^{\infty} \phi(n).$$

is a continuous, linear function, and thus satisfies the conditions of a distribution.

5. **Continuity of the derivative operator on test function spaces**

(a) If the sequence of smooth functions ϕ_j converges to 0 in $C^\infty(\mathbb{R})$, show that the sequence of derivatives ϕ_j' also converges to 0 in $C^\infty(\mathbb{R})$.

Solution. Start with the definition of convergence for a sequence of test functions in C^∞.

$$\begin{aligned} \lim_{j\to\infty} p_{m,k}(\phi_j) &= 0 \ \forall m, k = 0, 1, \ldots \\ \lim_{j\to\infty} p_{m,k}(\phi_j) &= \max_{\alpha\leq k} \sup_{||x||_2\leq m} |D^\alpha(\phi_j(x))| \end{aligned}$$

Now, given $\epsilon > 0$, $\exists M_{\epsilon,K+1}$ s.t. $m > M_{\epsilon,K+1}$,

$$\begin{aligned} \lim_{j\to\infty} p_{m,K+1}(\phi_j) &< \epsilon \\ (\implies) \lim_{j\to\infty} \max_{\alpha \le K+1} \sup_{||x||_2 \le m} |D^\alpha(\phi_j(x))| &< \epsilon. \end{aligned}$$

Now, consider the quantity

$$\begin{aligned} &\lim_{j\to\infty} \max_{\alpha+1 \le K+1} \sup_{||x||_2 \le m} |D^\alpha(\phi_j(x))| \\ &\qquad = \lim_{j\to\infty} \max_{\alpha \le K} \sup_{||x||_2 \le m} |D^\alpha(\phi_j'(x))| \\ &\qquad \le \lim_{j\to\infty} \max_{\alpha \le K+1} \sup_{||x||_2 \le m} |D^\alpha(\phi_j(x))| < \epsilon \end{aligned}$$

Therefore, given an $\epsilon > 0$, $\exists M_{\epsilon,K+1}$ s.t. $m > M_{\epsilon,K+1}$,

$$\lim_{j\to\infty} \max_{\alpha \le K} \sup_{||x||_2 \le m} |D^\alpha(\phi_j'(x))| < \epsilon$$

and so

$$\phi_j' \to 0 \in C^\infty.$$

(b) If the sequence of smooth functions ϕ_j converges to 0 in $\mathcal{S}(\mathbb{R})$, show that the sequence of derivatives ϕ_j' also converges to 0 in $\mathcal{S}(\mathbb{R})$.

Solution. Start with the definition of convergence for a sequence of test functions in $\mathcal{S}$.

$$\begin{aligned} \lim_{j\to\infty} q_{m,k}(\phi_j) &= 0 \ \forall m, k = 0, 1, \ldots \\ \lim_{j\to\infty} q_{m,k}(\phi_j) &= \max_{\alpha \le k} \sup_{||x||_2 \in \mathbb{R}} |x|^m |D^\alpha(\phi_j(x))| \end{aligned}$$

Now, given $\epsilon > 0$, $\exists M_{\epsilon,K+1}$ s.t. $m > M_{\epsilon,K+1}$,

$$\begin{aligned} \lim_{j\to\infty} q_{m,K+1}(\phi_j) &< \epsilon \\ (\implies) \lim_{j\to\infty} \max_{\alpha \le K+1} \sup_{||x||_2 \in \mathbb{R}} |x|^m |D^\alpha(\phi_j(x))| &< \epsilon. \end{aligned}$$

Now, consider the quantity

$$\begin{aligned} &\lim_{j\to\infty} \max_{\alpha+1 \le K+1} \sup_{||x||_2 \in \mathbb{R}} |x|^m |D^\alpha(\phi_j(x))| \\ &\qquad = \lim_{j\to\infty} \max_{\alpha \le K} \sup_{||x||_2 \in \mathbb{R}} |x|^m |D^\alpha(\phi_j'(x))| \\ &\qquad \le \lim_{j\to\infty} \max_{\alpha \le K+1} \sup_{||x||_2 \in \mathbb{R}} |x|^m |D^\alpha(\phi_j(x))| < \epsilon \end{aligned}$$

Therefore, given an $\epsilon > 0$, $\exists M_{\epsilon,K+1}$ s.t. $m > M_{\epsilon,K+1}$,

$$\lim_{j\to\infty} \max_{\alpha\leq K} \sup_{||x||_2\in\mathbb{R}} |D^\alpha(\phi_j'(x))| < \epsilon$$

and so

$$\phi_j' \to 0 \in \mathcal{S}.$$

6. **The translation operator**

Let χ denote any of the usual test function spaces. Define the translation operator τ_y on a smooth function ϕ by $\tau_y(\phi) = \psi \iff \psi(x) = \phi(x-y)$.

In all subsequent exercises, take χ to be $\mathcal{S}(\mathbb{R}^n)$.

(a) For any given $a \in \mathbb{R}^n$, show that τ_a is a continuous linear map from χ into itself.

Solution. Show linearity first, since it is easy.

$$\tau_a(\phi+\psi) = [\phi+\psi](x-a) = \phi(x-a) + \psi(x-a) = \tau_a(\phi) + \tau_a(\psi)$$

so the mapping is linear.

For continuity, assume the sequence $\phi_j \to 0$. Fix $m, k \in \mathbb{N}$.

$$\begin{aligned} q_{m,k}(\tau_a(\phi_j)) &= \max_{|\alpha|\leq k} \sup_{||x||\in\mathbb{R}^n} |x|^m |D^\alpha(\phi_j(x-a))| \\ &= \max_{|\alpha|\leq k} \sup_{||x||\in\mathbb{R}^n} ||x-a+a||^m |D^\alpha(\phi_j(x-a))| \\ &\leq \max_{|\alpha|\leq k} \sup_{||x||\in\mathbb{R}^n} \big(||x-a|| + ||a||\big)^m |D^\alpha(\phi_j(x-a))| \\ &= \sum_{i=0}^{m} C_i q_{m,k}(\phi_j) \to 0 \end{aligned}$$

where the last line utilizes a polynomial-like expansion. Therefore, $\phi_j \to 0 \implies \tau_a(\phi_j) \to \tau_a(0) = 0$. So, the mapping is continuous.

(b) For any given $\phi \in \chi$, show that the map $y \mapsto \tau_y(\phi)$ is continuous, and for any fixed $x \in \mathbb{R}^n$, the map $y \mapsto [\tau_y(\phi)](x)$ is also in χ.

Solution. Let $\chi = \mathcal{S}(\mathbb{R}^n)$. Assume $y_j \to 0$. Then

$$\begin{aligned} &\max_{|\alpha|\leq k} \sup_{x\in\mathbb{R}^n} ||x||^m \left|D^\alpha[\phi(x-y_j) - \phi(x)]\right| \\ &= \max_{|\alpha|\leq k} \sup_{x\in\mathbb{R}^n} ||x||^m \left|D^\alpha\phi(x-y_j) - D^\alpha\phi(x)\right| \\ &\leq \max_{|\alpha|\leq k} \sup_{x\in\mathbb{R}^n} ||x||^m \sup_{[x-y_j,x]} \left|D^{\alpha+1}\phi(x)\right| \cdot ||y_j|| \text{ (by the Mean Value Theorem)} \\ &\leq \max_{|\alpha|\leq k} \sup_{x\in\mathbb{R}^n} ||x||^m \left|D^{\alpha+1}\phi(x)\right| \cdot ||y_j|| \\ &\leq ||y_j|| \cdot q_{m,k+1}(\phi) \\ &\to 0 \text{ (since } y_j \to 0 \text{ and } q_{m,k+1}(\phi) \text{ is bounded).} \end{aligned}$$

Thus, the map $y \mapsto \tau_y(\phi)$ is bounded, and so it is continuous.

Now fix $x \in \mathbb{R}^n$. Then

$$\begin{aligned} q_{m,k}(\phi(x-y)) &= \max_{|\alpha|\le k} \sup_{y\in\mathbb{R}^n} ||y||^m \left|D^\alpha \phi(x-y)\right| \\ &= \max_{|\alpha|\le k} \sup_{(x-u)\in\mathbb{R}^n} ||x-u||^m \left|D^\alpha \phi(u)\right| \text{ (letting } u = x-y) \\ &\le \max_{|\alpha|\le k} \sup_{(x-u)\in\mathbb{R}^n} \sum_{j=0}^{m} \left(\begin{array}{c} m \\ j \end{array} \right) ||x||^{m-j} ||u||^j \left|D^\alpha \phi(u)\right| \text{ (letting } u = x-y) \\ &\le \sum_{j=0}^{m} ||x||^{m-j} q_{m-j,k}(\phi) \\ &< \infty \text{ (since } \phi \in \chi \text{ and } x \text{ is fixed).} \end{aligned}$$

Thus, the map $y \mapsto [\tau_y(\phi)](x)$ is also in χ.

The proofs are similar for $\chi = C_0^\infty$ and $\chi = C^\infty$ in that the first part uses the mean value theorem. The details are easier since we don't have to estimate term of the form $\|x\|^m$.

Let $\chi = C^\infty(\mathbb{R}^n)$. Assume $y_j \to 0$ and W.L.O.G $\|y_j\| \le 1$. Then, for any given $m \in \mathbb{N}$,

$$\begin{aligned} &\max_{|\alpha|\le k} \sup_{\|x\|\le m} \left|D^\alpha[\phi(x-y_j) - \phi(x)]\right| \\ &= \max_{|\alpha|\le k} \sup_{\|x\|\le m} \left|D^\alpha \phi(x-y_j) - D^\alpha \phi(x)\right| \\ &\le \max_{|\alpha|\le k} \sup_{\|x\|\le m+1} \left|D^{\alpha+1}\phi(x)\right| \cdot ||y_j|| \text{ (by the Mean Value Theorem)} \\ &\le ||y_j|| \cdot p_{m+1,k+1}(\phi) \\ &\to 0 \text{ (since } y_j \to 0 \text{ and } p_{m+1,k+1}(\phi) \text{ is bounded).} \end{aligned}$$

Thus, the map $y \mapsto \tau_y(\phi)$ is bounded, and so it is continuous. The same argument also works in $C_0^\infty(\mathbb{R}^n)$ once we recognize that, if $\mathrm{supp}(\text{Œ}) \subseteq \{\|\mathrm{x}\| \le \mathrm{R}\}$, then for all $j, \mathrm{supp}(\text{Œ}(. - \mathrm{y_j})) \subseteq \{\|\mathrm{x}\| \le \mathrm{L}\}$, where $L = \sup_j \|y_j\| < \infty$.

Now fix $x \in \mathbb{R}^n$. It is clear then $\phi(x-y) = \tau_{-x}(\psi(y))$ where $\psi(y) = -\phi(y)$. If ϕ is smooth, then, ψ is also smooth. If ϕ is compactly supported, then so is ψ. Finally, the results from part (a) show that translation by $-x$ preserves these properties.

(c) If $T : \chi \to \mathbb{R}$ is a distribution, show that the operator $\tau_y^*[T]$ defined by $\tau_y^*[T](\phi) = T(\tau_{-y}(\phi))$ is also a distribution.

Solution. Assume T is a distribution, that is the mapping defined by T is linear and continuous. To show that the operator $\tau_y^*[T](\phi)$ is a distribution, I need to show linearity and continuity.

To show that the operator is linear, consider

$$T(\tau_{-y}(\phi+\psi)) = T(\tau_{-y}(\phi) + \tau_{-y}(\psi)) = T(\tau_{-y}(\phi)) + T(\tau_{-y}(\psi))$$

by linearity of T. So, the operator is linear.

Now, we need continuity. Assume $\phi(x+y) = \tau_{-y}(\phi) \to 0$. By continuity of T, $T(\tau_{-y}(\phi)) \to T(0)$.

$\therefore$ the given operator is a distribution.

(d) If the distribution T is obtained from a function F with the usual identification $F \mapsto T_F$, then $\tau_y^*[T_F] = T_{\tau_y(F)}$.

Solution. Start with the definition of the mapping $F \mapsto T_F$,

$$T_F(\phi) = \int_{-\infty}^{\infty} F(x)\phi(x)dx.$$

Now, follow through the calculations,

$$\begin{aligned} T_F(\tau_{-y}(\phi)) &= T_F(\phi(x+y)) \\ &= \int_{-\infty}^{\infty} F(x)\phi(x+y)dx \end{aligned}$$

Now make a variable substitution in the integral to get,

$$\begin{aligned} T_F(\tau_{-y}(\phi)) &= \int_{-\infty}^{\infty} F(u-y)\phi(u)du \\ &= T_{\tau_y(F)}(\phi) \end{aligned}$$

Therefore, $\tau_y^*[T_F] = T_{\tau_y(F)}$.

7. Show that the mapping T_j defined by

$$T_j(\varphi) = \int_{\mathbb{R}} \min(j, -\log(|x|))\varphi(x)dx$$

is a distribution on the Schwartz space $\mathcal{S}(\mathbb{R})$.

Show that the sequence T_j converges in the sense of distributions on $\mathcal{S}(\mathbb{R})'$ and compute the derivative of the limit distribution.

Solution. Show that the sequence T_j converges in the sense of distributions on $\mathcal{S}(\mathbb{R})'$ and compute the derivative of the limit distribution.

So to prove that T_j is a distribution we first note that it is clearly linear and to prove continuity consider $\varphi_i \to 0$ in the sense of $S(\mathbb{R})$. Note that $\min(j, -\log|x|) = -\log|x|$ for all $|x| \geq e^{-j}$. For all x in magnitude less than e^{-j} the minimum is clearly j. Therefore we write for a fixed j,

$$T_j(\varphi_i) = \int_{|x|\leq e^{-j}} j\varphi_i dx - \int_{e^{-j}<|x|<1} \log|x|\varphi_i dx - \int_{|x|\geq 1} \log|x|\varphi_i dx$$

We bound the above and introduce the absolute value bars into the integrand,

$$|T_j(\varphi_i)| \leq \int_{|x|\leq e^{-j}} j|\varphi_i|dx + \int_{e^{-j}<|x|<1} |\log|x|||\varphi_i|dx + \int_{|x|\geq 1} |\log|x|||\varphi_i|dx$$

We bound the $|\varphi_i(x)| \leq q_{0,0}(\varphi_i)$, and also we use that $|\log|x|| < |x|$ for all $|x| > 1$. In the middle integral we use that the integrand is continuous so,

$$|T_j(\varphi_i)| \leq 2je^{-j}q_{0,0}(\varphi_i) + q_{0,0}(\varphi_i)\int_{e^{-j}<|x|<1} |\log|x||dx + \int_{|x|\geq 1} |x||\varphi_i|dx$$

In the second integral over $\log|x|$ we note that the integral is over a finite interval bounded away from 0. Finally use that

$$\int_{|x|\geq 1} |x||\varphi_i|dx = \int_{|x|\geq 1} |x|^3|\varphi_i|/|x|^2dx \leq q_{3,0}(\varphi_i)\int_{|x|>1} 1/|x|^2dx = 2q_{3,0}(\varphi_i)$$

Therefore each term above goes to zero since $\varphi_i \to 0$ implies that $q_{m,k}(\varphi_i) \to 0$ for all m, k. Therefore each T_j is a distribution.

Now let's posit the following limit for the T_j distribution, for any $\varphi \in S(\mathbb{R})$,

$$T(\varphi) = -\int_{\mathbb{R}} \log(|x|)\varphi(x)dx$$

First we show that is the limit in the sense of distributions. Therefore consider the difference,

$$|T_j(\varphi) - T(\varphi)| \leq \left|\int_{|x|\leq e^{-j}} j\varphi_i(x)dx - \left(-\int_{|x|\leq e^{-j}} \log(|x|)|\varphi_i(x)|dx\right)\right.$$
$$\left|\leq q_{0,0}(\varphi)\int_{|x|\leq e^{-j}} |j - [-\log|x|]|dx\right.$$

Since the integral is eve, it is clear that

$$\int_{|x|\leq e^{-j}} |j - [-\log|x|]|dx = 2\int_{0<x<1} |j - (-\log(|x|)|dx$$

Now consider that for the $-\log(|x|)$ is continuous away from zero, so we use this continuity such that for the positive integral, $|x - e^{-j}| < \eta$, $|j - (-\log(|x|)| = |-\log|e^{-j}| - (-\log(|x|)| \leq \epsilon$. The bound imposed on x can also be written $e^{-j} - \eta < x < e^{-j} + \eta$, and since the positive side of integral has $x < e^{-j}$ then this requirement is met.

$$|T(\varphi) - T_j(\varphi)| \leq q_{0,0}(\varphi)\epsilon$$

and we can make this as small as we want so the limit is obtained that $T_j(\varphi) \to T(\varphi)$ so the distribution converges in the distributional sense.

But is T a distribution? It is at least linear so Consider that for $|x| \leq 1$, that the $-\log|x|$ is bounded by the function $|x|^{-1/2}$ which is certainly integrable, i.e. for $\varphi_j \in S(\mathbb{R}) \to 0$

$$|T(\varphi_j)| \leq \int_{|x|\leq 1} |-\log(|x|)||\varphi_j|dx + \int_{|x|\geq 1} |-\log(|x|)||\varphi_j|dx$$

as we saw earlier the upper part of this integral is certainly well behaved. So we use the same procedure as before to find

$$|T(\varphi_j)| \leq q_{0,0}(\varphi_j) \int_{|x|\leq 1} |-\log(|x|)|dx + 2q_{3,0}(\varphi_j)$$

As we stated earlier,

$$\int_{|x|\leq 1} |-\log(|x|)|dx \leq 2\int_{0<x<1} x^{-1/2}dx = 2x^{1/2}|_{x=0}^{x=1} = 2$$

So clearly $T(\varphi_j) \to 0$ since the seminorms go to zero as $j \to \infty$. Therefore T the limit of T_j is a distribution in $S(\mathbb{R})'$.

To compute the distributional derivative of T, we use

$$T'(\phi) = \lim_j T_j'(\phi) = -\lim_j T_j(\phi')$$

We have, setting $\epsilon = e^{-j}$

$$\begin{aligned} T_j(\phi') &= \int_{|x|>\epsilon} -\log|x|\phi'(x)dx + j\int_{-e^{-j}}^{e^{-j}} \phi'(x)dx \\ &= \phi(x)\log|x||_{-\epsilon}^{\epsilon} + \int_{|x|>\epsilon} \frac{\phi(x)}{x}\,dx - \log(\epsilon)(\phi(\epsilon) - \phi(-\epsilon)) \\ &= \int_{|x|>e^{-j}} \frac{\phi(x)}{x}\,dx \end{aligned}$$

Taking the limit $j \to \infty$ proves that $T' = PV\frac{1}{x}$.